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A NOTE CONCERNING MAXIMA AND MINIMA OF FUNCTIONS OF SEVERAL VARIABLES.*

By DR. GILBERT AMES BLISS.

In the discussions of maxima and minima of functions of a single variable, it is usually assumed that the function $f(x)$ under consideration has n continuous derivatives in the neighborhood of a point a at which

$$f'(a)=f''(a)=\dots=f^{(n-1)}(a)=0 \text{ while } f^{(n)}(a)\neq 0.\dagger$$

The proofs usually given of the corresponding theorems for functions of several variables demand the existence and continuity of order $n+1$, that is of an order one higher than that of the first derivatives which do not vanish at the point under investigation.‡ In the following paragraphs a method is given for functions of several variables in which only the derivatives of the first n orders are used, as in the case of a single independent variable.

Suppose that all the derivatives of the function $f(x_1, x_2, \dots, x_k)$ up to and including those of the nth order exist and are continuous in a vicinity

$$V(a): \qquad a_i-\delta \leqq x_i \leqq a_i+\delta \quad (i=1, 2, \dots, k),$$

of the point $(a_1, a_2, \dots, a_k)$. If the derivatives of order less than n all vanish at $(a_1, a_2, \dots, a_k)$ then by Taylor's formula

$$(1) \quad f(a_1+h_1, a_2+h_2, \dots, a_k+h_k)-f(a_1, a_2, \dots, a_k)$$

$$=\sum \frac{h_1^{\alpha_1}}{\alpha_1!}\cdot\frac{h_2^{\alpha_2}}{\alpha_2!}\cdots\frac{h_k^{\alpha_k}}{\alpha_k!}\cdot\frac{\partial^n f}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\dots\partial x_k^{\alpha_k}},$$

where the summation is for all sets of positive integers $\alpha_1, \alpha_2, \dots, \alpha_k$ whose sum is n, and the arguments of the derivatives of f are $a_i+\theta h_i$ $(i=1, 2, \dots, k)$, $0<\theta<1$. If the right member is divided by r^n, where

$$r=\sqrt{[h_1^2+h_2^2+\dots+h_k^2]},$$

it becomes a homogeneous quadratic form H_n of degree n in the k ratios $\eta_i=\frac{h_i}{r}$ which satisfy the equation

$$(2) \qquad \eta_1^2+\eta_2^2+\dots+\eta_k^2=1.$$

Equation (1) may therefore be written

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†See for example, Pierpont, *The Theory of Functions of Real Variables*, p. 318.

‡Pierpont, *loc. cit.*, p. 322, *ff.*

$$f(a_1+h_1,\ a_2+h_2,\ \ldots,\ a_k+h_k)-f(a_1,\ a_2,\ \ldots,\ a_k)=r^n H_n(\eta,\ a+\theta h),$$

where η, $a+\theta h$ stand for the rows of letters $(\eta_1, \eta_2, \ldots, \eta_k)$ and $(a_1+\theta h_1, a_2+\theta h_2, \ldots, a_k+\theta h_k)$, respectively.

When the values $a+\theta h$ in H_n are replaced by the values a and the η's are allowed to take any values satisfying equation (2), the quadratic form $H_n(\eta, a)$ belongs to one of the following classes:

a) It is indefinite, *i. e.*, there exist values of η for which it has opposite signs; or

b) It is semi-definite, *i. e.*, it does not take on opposite signs but there exist some values of η for which it vanishes; or

c) It is definite, *i. e.*, for all values of η it has the same sign.

If the form $H(\eta, a)$ is indefinite, the function f can not have an extreme at the point $(a_1, a_2, \ldots, a_k)$.

The proofs of this statement and also of the theorems below depend upon the fact that the function $H(\eta, a+h)$ is a continuous function for all values of η and for all values of $h_1, h_2, \ldots, h_k$ defining points in the vicinity $V(a)$. Suppose that η' is a set of values for which $H(\eta', a)$ is negative. On account of its continuity $H(\eta, a+h)$ will also be negative for all points $a+h$ in a certain neighborhood $V'(a)$. The equations $h'_i=r\eta'_i$ may now be taken as defining the values h', and $r>0$ can be restricted so that all the points $a+h'$ lie in $V'(a)$. Then it is evident that for the points $a+h'$ the expression (1) is negative. In a similar way, by taking values of η'' at which $H(\eta'', a)>0$, points $a+h''$ can be found in any neighborhood of a such that the difference in (1) is positive.

If the form $H_n(\eta, a)$ is definite, then the function f has a minimum at the point $(a_1, a_2, \ldots, a_k)$ when H_n is positive, and a maximum when it is negative.

If $H_n(\eta, a)$ is definite, a neighborhood $V'(a)$ can be found such that $H_n(\eta, a+h)$ is also different from zero for all sets of values η satisfying (2), and for all points $a+h$ in $V'(a)$. This follows because $H_n(\eta, a+h)$ is continuous, and positive over the closed set of points (η, h) of $2k$ dimensions which satisfy the equation (2) and $h=0$. Hence in $V'(a)$ the difference in (1) is positive or negative according as $H_n(\eta, a)$ is positive or negative.

The function $H_n(\eta, a)$ cannot vanish for all sets of values η unless the derivatives of f of order n are all zero at the point a. It follows that when n is odd, $H_n(\eta, a)$ is either identically zero or indefinite, for if η' represents values for which it is different from zero, $H(-\eta', a)=-H(\eta', a)$. It is possible, then, to state the preceding results in the following form:

A function $f(x_1, x_2, \ldots, x_k)$ is given which with its first n derivatives is continuous in a certain neighborhood $V(a)$ of the point $(a_1, a_2, \ldots, a_k)$. The derivatives of the first $n-1$ orders are zero at the point $(a_1, a_2, \ldots, a_k)$, while sum at least of order n are different from zero. Then if n is odd, or if n is even and the form

$$H_n(\eta, a)=\sum\frac{\eta_1{}^{\alpha_1}}{\alpha_1!}\cdot\frac{\eta_2{}^{\alpha_2}}{\alpha_2!}\cdots\frac{\eta_k{}^{\alpha k}}{\alpha_k!}\frac{\partial^n f}{\partial x_1{}^{\alpha_1}\partial x_2{}^{\alpha_2}\ldots\partial x_n{}^{\alpha n}}$$

is indefinite at the point $(a_1, a_2, \ldots, a_k)$, *the function* f *has neither a maximum nor a minimum at the point* $(a_1, a_2, \ldots, a_k)$. *If* n *is even and* $H_n(\eta, a)$ *is positive definite the function* f *has a minimum, while if* $H_n(\eta, a)$ *is negative definite* f *has a maximum.*

The case, η even and $H_n(\eta, a)$ semi-definite, as usual requires a more elaborate investigation.

PRINCETON, *December, 1906.*

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DIVIDING BY ZERO.

By DR. G. A. MILLER.

Since the seventh century of our era the Hindus have considered division by zero,[*] and in the twelfth century the noted Hindu mathematician and astronomer Bhaskara gave the rule that the quotient obtained by dividing a number by zero is not changed by adding even a large number to it or by substracting a large number from it. The same thought is expressed in modern times by "the quotient obtained by dividing a number which is not zero by zero is infinite." This rule is commonly understood to mean, explained by Krishna, a commentator of Bhaskara, that if we divide a given number which is not zero by a number which is small the quotient may be made to exceed any finite number if the divisor is made sufficiently small.

According to this interpretation $a/0$ need not mean the same thing as $\frac{a}{b-b}$. The matter may be made perfectly clear by adopting the notation used by Professor Pierpont in his recent work on the Theory of Functions. If a and b are two distinct numbers then there is an infinity of numbers which do not differ any more from a than b does. The totality of these numbers are said to form the domain of a, whose norm is $a \sim b = \rho$. This totality or aggregate of numbers is denoted by $D_\rho\,(a)$. It is sometimes desirable to exclude a from its domain. In this case the domain is said to be deleted and it is denoted by $D_\rho{}^*(a)$. In particular, $D_\rho{}^*(0)$ means all the numbers which differ from zero by not more than ρ, with the exception of zero itself. By making ρ sufficiently small we obtain the aggregate of numbers which are commonly considered when we think of the meaning of $a/0$,

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[*] Cf. Algebra with arithmetic and mensuration from the Sanscrit of Brahmegupta and Bhascara translated by H. T. Colebrooke. London, 1817, p. 137.